Matter-Wave Solitons in the Presence of Collisional Inhomogeneities: Perturbation theory and the impact of derivative terms

S. Middelkamp,[1, *] P.G. Kevrekidis,[2, †] D.J. Frantzeskakis,[3, ‡] and P. Schmelcher[1, 4, §]

[1] *Theoretische Chemie, Physikalisch-Chemisches Institut,*
Universität Heidelberg, INF 229, 69120 Heidelberg, Germany
[2] *Department of Mathematics and Statistics,*
University of Massachusetts, Amherst MA 01003-4515, USA
[3] *Department of Physics, University of Athens,*
Panepistimiopolis, Zografos, Athens 157 84, Greece
[4] *Physikalisches Institut, Universität Heidelberg,*
Philosophenweg 12, 69120 Heidelberg, Germany
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We study the dynamics of bright and dark matter-wave solitons in the presence of a spatially varying nonlinearity. When the spatial variation does not involve zero crossings, a transformation is used to bring the problem to a standard nonlinear Schrödinger form, but with two additional terms: an effective potential one and a non-potential term. We illustrate how to apply perturbation theory of dark and bright solitons to the transformed equations. We develop the general case, but primarily focus on the non-standard special case whereby the potential term vanishes, for an inverse square spatial dependence of the nonlinearity. In both cases of repulsive and attractive interactions, appropriate versions of the soliton perturbation theory are shown to accurately describe the soliton dynamics.

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I. INTRODUCTION

The experimental creation of atomic Bose-Einstein condensates (BECs) has been one of the most fundamental developments in quantum and atomic physics over the past two decades. The impressive progress in this field due to intense experimental and theoretical studies has been already summarized in various books [1, 2, 3] and reviews [4]. This progress has been, to a considerable extent, fueled by the fact that, in a mean-field picture, BECs can be described by a macroscopic wavefunction obeying the Gross-Pitaevskii (GP) equation, which is an equation of the nonlinear Schrödinger (NLS) type. In such a mean-field description, the effective nonlinearity (which is introduced by interatomic interactions) allows for studies of macroscopic nonlinear matter waves; in this respect it is important to note that bright matter-wave solitons in attractive BECs [5, 6, 7], as well as dark [8, 9, 10, 11, 12, 13, 14] and gap [15] matter wave solitons in repulsive BECs, have been observed in a series of experiments (see also the recent review [4]).

One of the remarkable possibilities arising in the physics of BECs is that the interatomic interactions (and, hence, the effective nonlinearity) can be manipulated by means of different types of temporally- or spatially-varying external potentials. More specifically, the s-wave scattering length (which is proportional to the nonlinearity coefficient in the GP equation) can be experimentally adjusted using either magnetic [16, 17] or optical Feshbach resonances [18] in a very broad range. The availability of these tools has led to a number of consecutive theoretical and experimental studies. For instance, the formation of bright matter-wave solitons and soliton trains of ^{7}Li [5, 6] and ^{85}Rb [7] atoms used a tuning of the interatomic interactions from repulsive to attractive. Also, this type of manipulations was instrumental in achieving the formation of molecular condensates [19], and the probing of the BEC-BCS crossover [20]. A parallel track of theoretical studies has explored the use of a time-dependent modulation of the nonlinearity coefficient to stabilize attractive higher-dimensional

BECs against collapse [21], or to create robust matter-wave breathers in lower-dimensional BECs [22]. More recently, the use of spatial variations of the nonlinearity to create so-called "collisionally inhomogeneous" environments has been proposed. In that regard, major developments included adiabatic compression of matter-waves [23, 24], Bloch oscillations of matter-wave solitons [23], atomic soliton emission and atom lasers [25], enhancement of transmittivity of matter-waves through barriers [26, 27], dynamical trapping of matter-wave solitons [26], stable condensates exhibiting both attractive and repulsive interatomic interactions [28] as well as the delocalization transition of matter waves [29]. Among the types of spatial variations of the nonlinearity that have been proposed, one can trace linear ones [23, 26], as well as parabolic [30], random [31], periodic [29, 32, 33, 34], and localized (step-like) [25, 35, 36] ones. On the mathematical side, a number of detailed studies [37, 38, 39] have appeared, addressing aspects such as the effect of a "nonlinear lattice potential" (i.e., a spatially periodic nonlinearity) on the stability of matter-wave solitons, and the interplay between drift and diffraction/blow-up instabilities. More recently, the interplay of nonlinear and linear potentials has been examined in both continuum [40] and discrete [41] settings (see also the recent work [42] and references therein).

Our aim in this work is to study the dynamics of matter-wave solitons in the presence of a spatially-dependent nonlinearity. We consider both dark solitons in repulsive BECs, as well as bright solitons in attractive BECs. In the case where the sign of the nonlinearity coefficient (hereafter referred to as $g(x)$) does not change, we first show that a change of variables can convert the spatially variable nonlinearity problem into a "regular" one where the nonlinearity has a constant prefactor. This transformation results in the emergence of two additional perturbation terms: one of them can be considered as an effective potential term (i.e., a spatially-dependent function multiplying the macroscopic wavefunction u), while the other one can not (it consists of a spatially-dependent function multiplying the *derivative* of the wavefunction $\partial_x u$, for an elongated BEC along the x-direction). We use this transformation as a starting point in order to develop perturbation theory for the soliton dynamics in the presence of $g(x)$ for the case of arbitrary $g(x)$. However our focus is on the case where $g(x)$ is such that the potential term completely vanishes. The reason for this selection is that it appears to be the less physically intuitive case (due to the derivative nature of the corresponding perturbation). Moreover, the effect of a perturbation induced by a "standard" potential term has been studied fairly extensively in the BEC context (see, e.g., [1, 2, 3, 4]), also in the particular case of collisionally inhomogeneous BECs (see, e.g., [23, 24, 26]).

Our investigation is structured as follows. In the next section, we give the general setting and analyze the relevant transformation. In section III, we focus on dark solitons, first providing the general theory, and then applying it to the particular case of interest. In section IV, we follow a similar path for the case of bright solitons. Finally, in section V, we summarize our findings and present our conclusions, as well as some interesting directions for future study.

II. PERTURBED GROSS-PITAEVSKII EQUATION AND THE DERIVATIVE-ONLY CASE

In this work we will restrict ourselves to an effective one-dimensional (1D) description accounting only for the longitudinal dynamics of the condensate. In the transverse directions the atoms should be tightly confined which can be realized by an isotropic harmonic potential with a trap frequency $\omega_\perp$ (associated with the harmonic oscillator length $a_\perp$). For $a_\perp$ small enough, one can regard the transversal dynamics as frozen (see chap. 1 in [3] and also [43, 44]), i.e., only the corresponding ground state is occupied. The longitudinal motion takes place in the x-direction and should not be confined. Then, the corresponding mean-field equation is given by

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\partial_x^2\Psi + g(x)|\Psi|^2\Psi \qquad (1)$$

where $\Psi(x,t)$ is the macroscopic wave function, m the atomic mass and $g(x) = g^{(3D)}(x)/2\pi a_\perp^2$ the effective 1D interaction coefficient. The parameter $g^{(3D)}(x) = 4\pi\hbar^2 a/m$ characterizes the two-particle interaction in 3D with the s-wave scattering length a. The latter is positive (negative) for repulsive (attractive) condensates consisting of, e.g., ^{23}Na (^{7}Li) atoms. The value of the scattering length can be tuned, as mentioned above, e.g., by use of magnetic Feshbach resonances [46]. In the vicinity of a magnetic Feshbach resonance the value of the scattering length depends on the value of an applied magnetic field. Thus, one can achieve a spatially dependent scattering length by applying

an inhomogeneous magnetic field yielding a collisionally inhomogeneous BEC. We now use suitable straightforward rescalings (see, e.g., [23]) and dimensionless units to express Eq. (1) in the following form:

$$i\partial_t \Psi = -\frac{1}{2}\partial_x^2 \Psi + s|g(x)||\Psi|^2\Psi, \tag{2}$$

where the coefficient $s = \text{sgn}(g) = \pm 1$ for attractive and repulsive condensates, respectively. Applying the transformation $\Psi = \frac{u}{\sqrt{g}}$ allows us to rewrite eq. (2) in the following way:

$$i\partial_t u = -\frac{1}{2}\partial_x^2 u + s|u|^2 u + \tilde{V}_{eff}(x)u - \sqrt{g}\partial_x \frac{1}{\sqrt{g}}\partial_x u, \tag{3}$$

with the effective potential term $\tilde{V}_{eff}(x) = -\frac{1}{2}\sqrt{g}\partial_x^2 \frac{1}{\sqrt{g}}$. Equation (3) can be written as the usual NLS equation (with a defocusing or focusing nonlinearity for $s = \pm 1$, respectively) with an external spatially-dependent perturbation $P[u(x,t);x]$, namely:

$$i\partial_t u + \frac{1}{2}\partial_x^2 u - s|u|^2 u = P[u(x,t);x]. \tag{4}$$

The perturbation can be expressed as $P[u(x,t);x] = P_L[u(x,t);x] + P_{NP}[u(x,t);x]$, i.e., it consists of a linear effective potential contribution $P_L[u(x,t);x] = \tilde{V}_{eff}(x)u(x,t)$, as well as of a non-potential perturbation of the form $P_{NP}[u(x,t);x] = -\sqrt{g}\partial_x \frac{1}{\sqrt{g}}\partial_x u$. In this work we are mainly interested in the effects of the less standard, non-potential type of perturbation. Therefore, we assume the collisionally inhomogeneous interaction to be of the form

$$g(x) = \frac{1}{(D + Cx)^2}, \tag{5}$$

with arbitrary constants C and D. For such a particular selection of $g(x)$, $\tilde{V}_{eff}(x)$ vanishes leading to the perturbation

$$P[u(x,t);x] = -\frac{C}{D + Cx}\partial_x u, \tag{6}$$

consisting only of the non-potential contribution in the right hand side. Thus we can investigate the pure effects of the latter, non-standard contribution (the effects of a standard linear potential have been studied fairly extensively; see e.g. [1, 2, 3, 4]). We choose $C = 1$ and $D = -200$; thereby the singularity in the perturbation occurs at $x_0^{sing} = 200$ (which will be outside the region of interest in our domain). For this choice, the spatial dependence of the coefficient $g(x)$ is shown in Fig. 1.
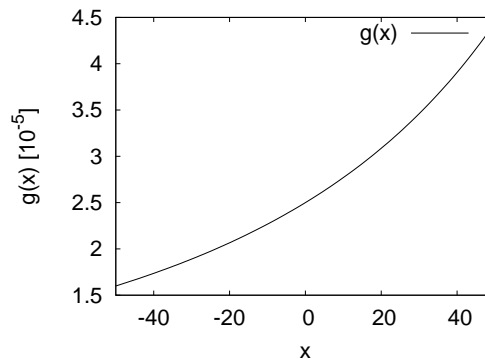

FIG. 1: Spatial dependence of the interaction parameter $g(x)$ for $C = 1$ and $D = -200$ for a repulsive condensate.

III.   DARK MATTER-WAVE SOLITONS

A.   Full perturbative approach

Let us first consider the case of dark matter-wave solitons for $s = 1$. In order to treat effects of the perturbation (6) on a dark soliton analytically we employ the adiabatic perturbation theory assuming that the functional form of the soliton remains unchanged by the perturbation (an assumption that in our setting will be justified a posteriori). We first use the transformation $u \to u \exp(-it)$ to put Eq. (4) in the form $i\partial_t u + \frac{1}{2}\partial_x^2 u - (|u|^2 - 1)u = P[u(x,t); x]$ and use as an ansatz for the soliton, the following expression,

$$u = B\tanh(B(x - x_0)) + iA, \tag{7}$$

which is the exact dark soliton solution of the above mentioned unperturbed NLS equation. According to the above discussion, the soliton depth A, velocity B (with $A^2 + B^2 = 1$), and center x_0 are assumed to be unknown functions of time. The Lagrangian density of an unperturbed dark soliton is given by [45]:

$$\mathcal{L}(u) = \frac{i}{2}(u^\star \partial_t u - u\partial_t u^\star)(1 - \frac{1}{|u|^2}) - \frac{1}{2}|\partial_x u|^2 - \frac{1}{2}(|u|^2 - 1)^2, \tag{8}$$

while the averaged Lagrangian, $L = \int dx \mathcal{L}(u)$, can be calculated by substituting (7) in Eq. (8) yielding:

$$L(A, x_0) = 2\partial_t x_0 \left(-AB + \tan^{-1}(\frac{B}{A})\right) - \frac{4}{3}B^3. \tag{9}$$

In [47] it was shown that, within the framework of the adiabatic perturbation theory for small perturbations, the parameters of the soliton $\alpha_j = \{x_0, A\}$ obey the the Euler-Lagrange equations

$$\partial_{\alpha_j} L - \frac{d}{dt}\partial_{\alpha_j'} L = 2Re\{\int dx P^\star(u)\partial_{\alpha_j} u\} \tag{10}$$

with $\alpha_j' = \partial_t \alpha_j$. This leads to a system of ordinary differential equations (ODE) for A and x_0:

$$\begin{aligned}
\partial_t A &= \frac{1}{2}B^3 \int dx\sqrt{g}\partial_x \frac{1}{\sqrt{g}}\text{sech}^4(B(x - x_0)) \\
&\quad + \frac{1}{4}B^2 \int dx\sqrt{g}\partial_x^2 \frac{1}{\sqrt{g}}\tanh(B(x - x_0))\text{sech}^2(B(x - x_0)), \tag{11} \\
\partial_t x_0 &= A - \frac{1}{2}A \int dx\sqrt{g}\partial_x \frac{1}{\sqrt{g}}\text{sech}^2(B(x - x_0))\Big(\tanh(B(x - x_0)) \\
&\quad + B(x - x_0)\text{sech}^2(B(x - x_0))\Big) \\
&\quad - \frac{1}{4}\int dx\sqrt{g}\partial_x^2 \frac{1}{\sqrt{g}}\Big((\frac{1}{B}\tanh^2(B(x - x_0)) - 1) \\
&\quad + (x - x_0)\tanh(B(x - x_0))\text{sech}^2(B(x - x_0))\Big). \tag{12}
\end{aligned}$$

For an interaction obeying Eq. (5), the terms arising from the linear potential vanish, leading to the system:

$$\begin{aligned}
\partial_t A &= \frac{1}{2}B^3 \int dx \frac{C}{D + Cx}\text{sech}^4(B(x - x_0)), \tag{13} \\
\partial_t x_0 &= A - \frac{1}{2}A \int dx \frac{C}{D + Cx}\text{sech}^2(B(x - x_0))\Big(\tanh(B(x - x_0)) \\
&\quad + B(x - x_0)\text{sech}^2(B(x - x_0))\Big). \tag{14}
\end{aligned}$$

B. Approximations

We can simplify the general framework of Eqs. (11,12) by performing a Taylor expansion of the interaction around $x = x_0$, leading in first order to

$$\partial_t A \approx \frac{2}{3}(1 - A^2)\sqrt{g(x_0)}\partial_x \frac{1}{\sqrt{g(x)}}\Big|_{x=x_0} \tag{15}$$

$$\partial_t x_0 \approx A + \frac{1}{4}\frac{A}{B^2}\sqrt{g(x_0)}\partial_x^2 \frac{1}{g(x)}\Big|_{x=x_0}. \tag{16}$$

The Taylor expansion around x_0 can be justified in most settings due to the exponential localization of the soliton around its center. Dropping higher-order terms essentially implies that the interaction does not change on the scale of the width of the soliton (if the latter assumption is invalid, then we can not resort to this approximation). In the special case of the interaction (5) the following evolution equations are obtained:

$$\partial_t A \approx \frac{2}{3}(1 - A^2)\frac{C}{D + Cx_0} \tag{17}$$

$$\partial_t x_0 \approx A \tag{18}$$

By combination of Eqs. (17) and (18), we obtain a single second-order ordinary differential equation (ODE)

$$\partial_t^2 x_0 = \frac{2}{3}\frac{C}{D + Cx_0}(1 - (\partial_t x_0)^2) \tag{19}$$

for the center of the soliton. If the velocity of the soliton is small, $\partial_t x_0 \ll 1$, one can neglect the second term in the right hand side of Eq. (19) leading to:

$$\partial_t^2 x_0 = \frac{2}{3}\frac{C}{D + Cx_0}. \tag{20}$$

We will discuss the validity of this approximation in our numerical results below. Equation (20) is the equation of motion (EOM)

$$\partial_t^2 x_0 = -\partial_{x_0}V^{eff}(x_0), \tag{21}$$

of a particle in the presence of the effective potential

$$V^{eff}(x_0) = -\frac{2}{3}\ln(|Cx_0 + D|). \tag{22}$$

Therefore, we will denote Eq. (19) as EOM and Eq. (20) as EOM_a in the next section. As an interesting aside, we should note that even in the presence of the kinetic term (i.e., if $(\partial_t x_0)^2$ is not neglected), one can rewrite Eq. (19) as a Hamiltonian system (see Ref. [48]) using a generalized momentum $P = g(x_0)\partial_t x_0$. With this momentum one finds for a system

$$\partial_t^2 x_0 = f(x_0)(1 - (\partial_t x_0)^2), \tag{23}$$

the equations of motion

$$\partial_t x_0 = \frac{P}{g(x_0)}, \tag{24}$$

$$\partial_t P = g(x_0)f(x_0) + P^2\Big(\frac{\partial_{x_0}g(x_0)}{(g(x_0))^2} - \frac{f(x_0)}{g(x_0)}\Big), \tag{25}$$

which correspond to the Hamiltonian

$$H(x_0, P) = \frac{1}{2}\frac{P^2}{g(x_0)} + F(x_0), \tag{26}$$

with

$$g(x_0) = A \exp\left(2 \int^{x_0} dx_0' f(x_0')\right), \tag{27}$$

$$F(x_0) = -\frac{1}{2} g(x_0). \tag{28}$$

In the particular case of $f(x_0) = \frac{2}{3} \frac{C}{D + C x_0}$, one obtains for the momentum

$$P = A(D + C x_0)^{\frac{4}{3}} \partial_t x_0, \tag{29}$$

and for the Hamiltonian

$$H(x_0, P) = \frac{P^2}{2A}(D + C x_0)^{-\frac{4}{3}} - \frac{1}{2} A(D + C x_0)^{\frac{4}{3}}. \tag{30}$$

C.   Numerical Results

In this section we present and compare the numerical results obtained by solving the full partial differential equation (PDE) of the GP type (4), as well as the ODEs (13,14), the EOM (19) and the simplified EOM_a (20). We have confirmed that throughout our simulations the soliton is localized in a region with a well defined perturbation avoiding the singularity. Since the soliton is exponentially localized, the spatial integrations in Eqs. (13,14) can be restricted to a region around the center of the soliton with a finite perturbation and a well defined integrand. The time evolution is performed by the Adams-Bashforth-Moulton predictor-corrector method.

Figure 2 shows the time evolution of the density profile of a dark soliton with $x_0(0) = 0$ for different initial velocities obtained by solving Eq. (4). Black represents the highest density, while white corresponds to the lowest density. The dotted lines are the results for x_0 (the center of the soliton) as obtained by solving the ODEs (13,14). The results agree very well, showing that the adiabatic perturbation theory describes the motion of the center of the soliton accurately. For $A_{init} = 0$ (lowest curve) the soliton gets accelerated to the negative half-plane and moves immediately into this direction. For $A_{init} = 0.25$ and $A_{init} = 0.5$ (middle and top curve, respectively), the soliton also gets accelerated into the direction of the negative half-plane but starts moving to the positive one due to its initial velocity until it reaches a turning point of zero velocity and changes direction. By investigation of Fig. 1 one observes that in the considered region the value of the interaction parameter decreases for decreasing x. Thus, the soliton gets accelerated into the direction with a smaller interaction parameter. Due to the fact that the interaction is repulsive, the interaction energy decreases with decreasing interaction parameter. So the soliton tends to move into the region with less interaction energy. This happens despite the fact that the interaction parameter does not enter explicitly in the perturbation as a potential but rather through the product of its first derivative and the first derivative of the scaled wavefunction.
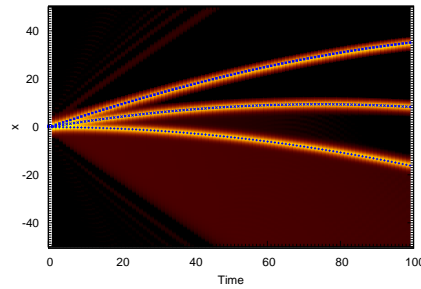

FIG. 2: Time evolution of the density of dark solitons with $x_0(0) = 0$ and $A_{init} = 0$, 0.25, 0.5 (from bottom to top). The dotted line is the CM parameter (x_0) obtained by solving the ODEs (13)-(14).

Figure 3 shows the time evolution for $x_0(0) = 0$, $A_{init} = 0$(a) and $A_{init} = 0.5$(b) of the differences between the results for the center of the soliton obtained by solving the PDE, and the ODE, EOM and EOM_a, respectively. We calculated the center of mass of the PDE solution by performing the integration $x_0 = \int x(b - |u|^2)dx / \int (b - |u|^2)dx$, with $b = |u(x_b)|^2$ being the background density evaluated far away from the center of the soliton. The differences for the ODEs and the EOM are almost equal and small for both initial velocities. So the adiabatic perturbation theory works fine for describing the center of the soliton. The results obtained by solving EOM_a coincide for a small time period with the result of the PDE. For longer times, they deviate from these results. For a larger initial velocity the deviation is even larger. The reason for this is that we neglected the impact of the velocity in EOM_a and, thus, the approximation gets worse for larger velocities. However, the qualitative behavior is described correctly even within this approximation. Hence, one can understand the behavior of the soliton as a particle moving in the effective potential (22) and thus can explain the acceleration observed in Fig. 2.
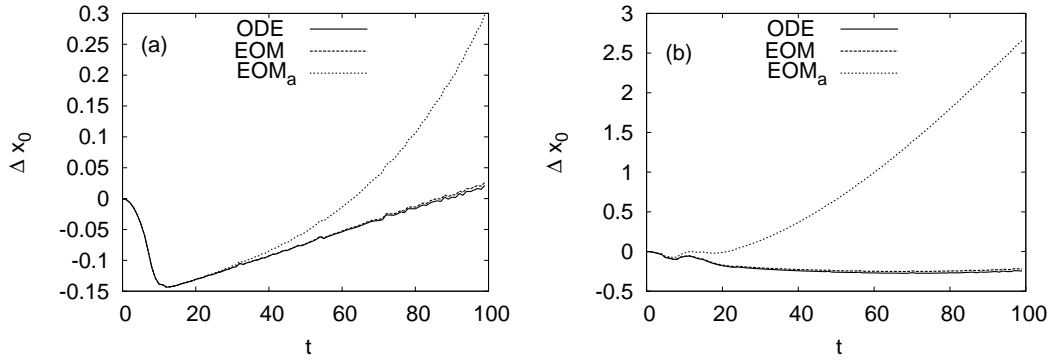

FIG. 3: Difference of the soliton center Δx_0 calculated by solving the PDE with the result of the ODEs, EOM and EOM_a for $A_{init} = 0.0$ (a) and $A_{init} = 0.5$ (b).

IV. BRIGHT MATTER-WAVE SOLITONS

A. Full perturbative approach

In the case of attractive interactions ($s = -1$), Eq. (4) reads after substitution of $\tau = t/2$

$$i\partial_\tau u + \partial_x^2 u + 2|u|^2 u = 2\epsilon P(u). \tag{31}$$

In the absence of perturbations, it is well known that Eq. (31) possesses a bright soliton solution of the form

$$u(z, t) = 2i\eta \exp(-2i\xi x - i\Phi)\text{sech}(z), \tag{32}$$

where $z = 2\eta(x - \zeta)$, while η represents the amplitude, Φ the phase and ζ the center of the soliton, and ξ is related to the velocity of the soliton. For a small perturbation, we can now employ the adiabatic perturbation theory for bright solitons [49] to treat the perturbation effects analytically. Then, the soliton parameters become slowly-varying functions of time, however the shape of the soliton remains unchanged (once again this is a principal assumption that will be justified a posteriori). With the general perturbation arising due to a spatially-dependent scattering length, one arrives at

the following system of ordinary differential equations for the parameters of the soliton:

$$\partial_\tau \eta = 8\eta^2 \xi \int dx \sqrt{g} \partial_x \frac{1}{\sqrt{g}} \text{sech}^2(2\eta(x-\zeta)), \tag{33}$$

$$\partial_\tau \xi = 8\eta^3 \int dx \sqrt{g} \partial_x \frac{1}{\sqrt{g}} \tanh^2(2\eta(x-\zeta)) \text{sech}^2(2\eta(x-\zeta))$$
$$- 2\eta^2 \int dx \sqrt{g} \partial_x^2 \frac{1}{\sqrt{g}} \tanh(2\eta(x-\zeta) \text{sech}^2(2\eta(x-\zeta)), \tag{34}$$

$$\partial_\tau \zeta = -4\xi + 8\eta \xi \int dx \sqrt{g} \partial_x \frac{1}{\sqrt{g}} (x-\zeta) \text{sech}^2(2\eta(x-\zeta)), \tag{35}$$

$$\partial_\tau \Phi = 4(\xi^2 - \eta^2) + 8\eta^2 \int dx \sqrt{g} \partial_x \frac{1}{\sqrt{g}} \text{sech}^2(2\eta(x-\zeta)) \tanh(2\eta(x-\zeta))$$
$$(1 - 2\eta x \tanh(2\eta(x-\zeta)))$$
$$- 2\eta \int dx \sqrt{g} \partial_x^2 \frac{1}{\sqrt{g}} \text{sech}^2(2\eta(x-\zeta)) \big(1 - 2\eta x \tanh(2\eta(x-\zeta))\big). \tag{36}$$

For an interaction parameter of the form (5) one obtains:

$$\partial_\tau \eta = 8\eta^2 \xi \int dx \frac{C}{D+Cx} \text{sech}^2(2\eta(x-\zeta)) \tag{37}$$

$$\partial_\tau \xi = 8\eta^3 \int dx \frac{C}{D+Cx} \tanh^2(2\eta(x-\zeta)) \text{sech}^2(2\eta(x-\zeta)) \tag{38}$$

$$\partial_\tau \zeta = -4\xi + 8\eta \xi \int dx \frac{C}{D+Cx} (x-\zeta) \text{sech}^2(2\eta(x-\zeta)) \tag{39}$$

$$\partial_\tau \Phi = 4(\xi^2 - \eta^2) + 8\eta^2 \int dx \frac{C}{D+Cx} \text{sech}^2(2\eta(x-\zeta)) \tanh(2\eta(x-\zeta))$$
$$(1 - 2\eta x \tanh(2\eta(x-\zeta))). \tag{40}$$

B. Approximations

From the above equations it is clear that Eq. (40) describes the time evolution of the phase of the soliton which, however, does not emerge in the equations determining the other parameters. Therefore, we will restrict our considerations to Eqs. (37)-(39). Since the soliton is exponentially localized around $x = \zeta$ we can perform a Taylor expansion around ζ and thus simplify Eqs. (33)-(35) as follows:

$$\partial_\tau \eta = 8\eta \xi \sqrt{g(\zeta)} \partial_x \frac{1}{\sqrt{g(x)}} \Big|_{x=\zeta} \tag{41}$$

$$\partial_\tau \xi = \frac{8}{3}\eta^2 \sqrt{g(\zeta)} \partial_x \frac{1}{\sqrt{g(x)}} \Big|_{x=\zeta} \tag{42}$$

$$\partial_\tau \zeta = -4\xi. \tag{43}$$

The physical interpretation of this approximation is that the interaction parameter does not vary over the width of the soliton. Focusing more specifically on a collisional inhomogeneity of the form of Eq. (5) where these contributions vanish exactly, leads to

$$\partial_\tau \eta = \eta \xi \frac{8C}{D+C\zeta} \tag{44}$$

$$\partial_\tau \xi = \frac{8}{3}\eta^2 \frac{C}{D+C\zeta} \tag{45}$$

$$\partial_\tau \zeta = -4\xi. \tag{46}$$

We can solve the simplified Eq. (44) directly by using Eq. (46):

$$\eta = \eta(0) \frac{(C\zeta(0)+D)^2}{(C\zeta+D)^2}. \tag{47}$$

Combination of Eqs. (45-47) and back transformation to the real time t leads to the equation of motion for the soliton center:

$$\partial_t^2 \zeta = -\frac{8}{3} C \eta(0)^2 \frac{(C\zeta(0) + D)^4}{(C\zeta + D)^5},$$ (48)

with an associated effective potential:

$$V^{eff}(\zeta) = \frac{2}{3} \eta(0)^2 \frac{(C\zeta(0) + D)^4}{(C\zeta + D)^4}.$$ (49)

C. Numerical Results

Figure 4 shows the time evolution of the density of the bright soliton (note that, in contrary to before, black represent zero density, while white represents high density). The results are obtained by integration of Eq. (4) with an initial state given by Eq. (32) with parameters initialized as $\eta = 0.5$, $\Phi = \zeta = 0$ and $\xi_{init} = 0$, 0.25, 0.5 (from top to bottom). The dotted line shows the corresponding results for the center of the soliton ζ obtained by solving Eqs. (37-40). The results of the perturbation theory once again agree very well with the results of the PDE. In the case of zero initial velocity (top curve), the soliton gets accelerated to the positive half-plane and starts moving into this direction immediately. For positive ξ_{init} the initial velocity is negative, according to Eq. (38), leading to a motion towards the negative half-plane. However, the soliton still moves toward the direction of the positive half-plane, due to its initial speed, yet eventually it acquires a zero velocity and a change of the direction of motion occurs. The direction of the acceleration is the direction of increasing interaction parameter as can be seen by comparing the results with Fig. 1, as this minimizes the energy of the system (even though the interaction does not act, strictly speaking, as a potential).
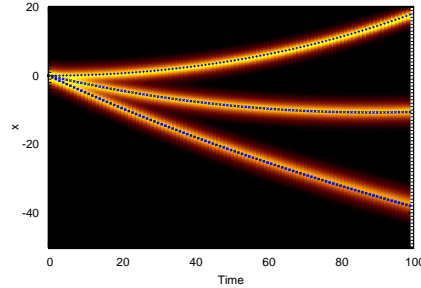

FIG. 4: Time evolution of the density of a bright soliton with initial parameter $\eta = 0.5$, $\Phi = \zeta = 0$ and $\xi_{init} = 0$, 0.25, 0.5 (from top to bottom). The dotted line shows the result for ζ (the center of the soliton) from the adiabatic perturbation theory.

Figure 5 shows the difference of the center of the soliton calculated by the PDE with the results of the ODEs (37)-(39) and the EOM (48). The center of the soliton of the PDE solution is determined by the quotient $\zeta = \int x|u|^2 dx / \int |u|^2 dx$. Figure 5a shows the differences for $\eta = 0.5$, $\Phi = \zeta = 0$ and $\xi_{init} = 0$. The difference of the EOM result from the PDE result is slightly larger than the difference of the ODE result. Both differences increase with time but they are still very small for the time period considered. Fig. 5b shows the differences for $\eta = 0.5$, $\Phi = \zeta = 0$ and $\xi_{init} = 0.5$. In this case, the absolute differences are an order of magnitude larger than in the previous case. However, compared to the position and width of the soliton one can still regard them as small. The results of the ODEs and the EOM are almost equal. A conclusion of this investigation is that the dynamics of the soliton is described fairly accurately by the model of a particle subject to the effective potential (49).
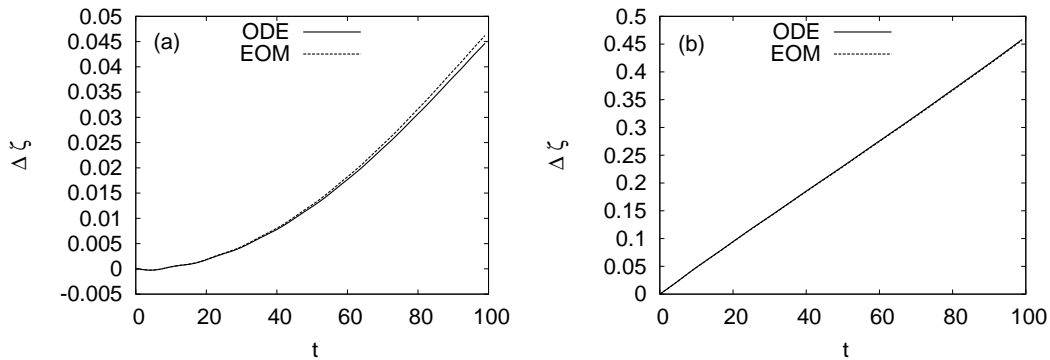
FIG. 5: Difference of the soliton center calculated by solving the ODE and EOM to the result of the PDE with initial parameter $\eta = 0.5$, $\Phi = \zeta = 0$ and $\xi_{init} = 0$ (a) or $\xi_{init} = 0.5$ (b).

V.   CONCLUSIONS AND FUTURE CHALLENGES

In this work, we considered the effect of (slowly-varying) spatially dependent nonlinearities of a definite sign on both dark and bright matter-wave solitons of repulsive and attractive Bose-Einstein condensates, respectively. We have shown that a relevant transformation can be employed to convert the spatially dependent nonlinear problem into one of spatially uniform nonlinearity, at the expense of introducing two perturbative terms. One of the latter is in the form of a linear potential (which have been considered extensively previously), while the other constitutes a non-potential type of perturbation, being proportional to the spatial derivative of the field. To especially highlight the non-potential nature of the second term, we considered collisional inhomogeneities of inverse square spatial dependence, whereby the linear potential perturbation identically vanishes, and the purely non-potential one has to be considered. Even in these settings (but also more generally), we found that soliton perturbation theory provides a powerful tool towards describing such collisional inhomogeneities.

It would be interesting to extend the present considerations in a number of directions. Firstly, it would be relevant to appreciate the effect of $g(x)$ on higher-dimensional structures, such as vortices, and on their stability. On the other hand, it would be especially interesting even in one spatial dimension to determine whether techniques like the ones used here (or variants thereof) can be applied to cases where the sign of the nonlinearity changes. Finally, it would be relevant to observe systematically how techniques such as soliton perturbation theory may fail, as the size of spatial extent of the collisional inhomogeneity becomes comparable to that of the solitary wave and to understand the ensuing phenomenology in such cases. Studies along some of these directions are currently in progress and will be reported in future publications.

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